

On application of involutivity analysis of differential equations to constrained dynamical systems ^{*}

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A brief sketch of computer methods of involutivity analysis of differential equations is presented in context of its application to study degenerate Lagrangian systems. We exemplify the approach by a detailed consideration of a finite-dimensional model, the so-called light-cone $SU(2)$ Yang-Mills mechanics. All algorithms are realized in computer algebra system *Maple*.

1 Introduction

Among the properties of systems of analytical partial differential equations (PDEs) which can be studied without explicit integration there are two important ones: the question of their compatibility and the problem of posing of an initial-value problem (Cauchy problem), providing the existence and the uniqueness of an analytical solution. Both of these problems are crucial for the correct formulation of the evolution of degenerate Lagrangian dynamical systems.

The main obstacle in the study of these problems for PDEs of a given order q is the existence of “hidden” *integrability conditions*. These conditions, $q' \leq q$ order differential equations, are consequences of the given system of PDEs that can not be derived using only algebraic manipulations with the PDEs. The special class of PDEs, called an *involutive*

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system of PDEs, has all such integrability conditions incorporated in it. This means that differentiation of the system, called *prolongation*, do not reveal new integrability conditions. Examples of such involutive systems are the quasilinear systems of Cauchy-Kowalevskaya type (normal systems).

The extension of a system by its integrability conditions is called a completion. From the completion point of view the linear homogeneous systems of PDEs with constant coefficients can be associated with systems of pure polynomial equations [1]-[5]. The polynomial involutive systems [4] provide a fruitful algorithmic tool in commutative algebra [6].

The general algorithmic foundation of the involutive approach is based on the concept of involutive monomial division invented in [4] and defined for a finite monomial set. Every particular division provides for each monomial in the set the self-consistent separation of variables into multiplicative and non-multiplicative ones. In the case of linear differential system all its integrability conditions can be constructed by sequential performing of multiplicative reductions for non-multiplicative prolongations of the equations in the system [5].

The polynomial and linear differential involutive systems generate *involutive bases* of polynomial ideals [4] and linear differential ideals [5], respectively. They are polynomial [7, 8] and differential Gröbner bases [9, 10] of special form. Though the involutive bases are generally redundant as the Gröbner ones, their use make more accessible the structural information of the polynomial and differential ideals. The Janet bases [4] may be cited as typical representatives of involutive bases and have been used in algebraic and Lie symmetry analysis of differential equations [5].

The completion of differential equations to involution is the most universal algorithmic method for their algebraic analysis [5] and can be applied for the following purposes:

- Check the compatibility of the systems of PDEs. In case of system inconsistency there is an integrability condition of the form $1 = 0$ which is revealed in the course of completion.
- Detection of arbitrariness in the general analytic solution of analytic systems of PDEs [2, 5, 13].
- Elimination of a subset of dependent variables, that is, obtaining differential consequences of the given system, if they exist, which do not contain the dependent variables specified.
- Posing of an initial value problem for a system of analytic PDEs providing existence and uniqueness of locally holomorphic solutions [2, 5].
- Lie symmetry analysis of differential equations. Completion to involution of the determining equations for the Lie symmetry generators is the most general algorithmic method of their integration [5, 14].

- Preprocessing for numerical integration. Certain non-commutative involutive bases can be used for automatic generation of finite-difference schemes PDEs [15, 16].
- Computation of the complete set of constraints for degenerated dynamical systems and their separation into first and second classes [17, 18, 19].

Below the last application will be exemplified by studying the finite-dimensional degenerate system, the so-called light-cone Yang-Mills mechanics. In order to make presentation more transparent we start with a very short introduction of the main settings of the involutive method.

2 Involutive polynomial bases

The basic algorithmic ideas go back to M. Janet [1] who invented the constructive approach to study of PDEs in terms of the corresponding monomial sets based on the following association between derivatives and monomials:

$$\frac{\partial^{\mu_1 + \dots + \mu_n} u^\alpha}{\partial x_1^{\mu_1} \dots \partial x_n^{\mu_n}} \iff [x_1^{\mu_1} \dots x_n^{\mu_n}]_\alpha. \quad (1)$$

Note that the monomials associated to different dependent variables u^α belong to different monomial sets.

The association (1) allows to reduce the problem of involutivity analysis of linear homogeneous systems of PDEs to the same problem for pure algebraic systems [1]-[3],[5]. Having in mind this fact we shall state now the main definitions and results concerning the involutivity of algebraic systems.

Let $R = K[x_1, \dots, x_n]$ be a ring of multivariate polynomials over a zero characteristic coefficient field K . Then a finite set $F = \{f_1, \dots, f_m\} \subset R$ of polynomials in R is a *basis of the ideal*

$$\langle F \rangle = \langle f_1, \dots, f_m \rangle = \left\{ \sum_{i=1}^m h_i f_i \mid h_j \in R \right\}. \quad (2)$$

In the involutive approach to commutative (polynomial) algebra [4], which is a mapping of the involutivity analysis of linear PDEs [2, 5], for every polynomial in the finite set F the set variables x_1, \dots, x_n are separated into disjoint subsets of *multiplicative* and *nonmultiplicative* variables.

To be self-consistent such a separation must satisfy some axioms [4] and every appropriate separation generates an *involutive monomial division* in the following sense. Fix a linear *admissible monomial order* \succ such that

$$m \neq 1 \implies m \succ 1, \quad (3)$$

$$m_1 \succ m_2 \iff m_1 m \succ m_2 m \quad (4)$$

holds for any monomials (power products of the variables with integer exponents) m, m_1, m_2 . Then for every polynomial f in F one can select its *leading monomial* $lm(f)$ (with

respect to \succ). All leading monomials in F form a finite monomial set U . If $u \in U$ divides a monomial w such that all the variables which occur in w/u are multiplicative for u , then u is called *involutive divisor* of w . We shall denote by L an involutive division, which specifies a set of multiplicative (resp. nonmultiplicative) variables for every monomial u in any given finite monomial set U and write $u|_L w$ if u is $(L-)$ involutive divisor of w . In the latter case we shall also write $w = u \times v$ where, by the above definition, the monomial $v = w/u$ contains only multiplicative variables.

In the papers [4, 11, 12] several involutive divisions were introduced and studied in detail. Here, as an example we present one called after M. Janet, who was one of the founders of the involutive approach to PDEs and who devised the related separation of variables [1].

Given a finite set U of monomials in $\{x_1, \dots, x_n\}$ and a monomial $u = x_1^{d_1} \cdots x_n^{d_n} \in U$, a variable x_i ($i > 1$) is Janet multiplicative for u if its degree d_i in u is maximal among all the monomials in U having the same degrees in the variables x_1, \dots, x_{i-1} . As for x_1 , it is *Janet multiplicative* for u if d_1 takes the maximal value among the degrees in x_1 of monomials in U . If a variable is not Janet multiplicative for u in U it is considered as *Janet nonmultiplicative*.

Consider, for example, a monomial set

$$U = \{x_1 x_2, x_2 x_3, x_3^2\}. \quad (5)$$

This gives the following Janet multiplicative and nonmultiplicative variables for monomials in U :

Monomial	Variables	
	Multiplicative	Nonmultiplicative
$x_1 x_2$	x_1, x_2, x_3	—
$x_2 x_3$	x_2, x_3	x_1
x_3^2	x_3	x_1, x_2

Given a finite polynomial set F , a noetherian [4] involutive division L , for instance, Janet division, and an admissible monomial order \succ , one can algorithmically construct [4] a *minimal L -involutive basis* or L -basis $G \subset R$ of the ideal $\langle F \rangle = \langle G \rangle$ such that for any polynomial f in the ideal there is a polynomial g in G satisfying $lm(g)|_L lm(f)$, and every polynomial g in G does not contain monomials having involutive divisors among the leading monomials of other polynomials in G .

If $F = \{f_1, \dots, f_m\} \subset R$ is a polynomial set, L is an involutive division and \succ is an admissible monomial order, then any polynomial p in R can be rewritten (reduced) modulo the ideal $\langle F \rangle$ as

$$p = h - \sum_{i=1}^m \sum_j a_{ij} f_i \times u_{ij}, \quad (6)$$

where a_{ij} are elements (coefficients) of the base field K , u_{ij} are L -multiplicative monomials for $lm(f_i)$ such that $lm(f)u_{ij} \preceq lm(p)$ for all i, j , and there are no monomials occurring in h which have L -involutive divisors among $\{lm(f_1), \dots, lm(f_m)\}$. In this case h is said to be in the L -normal form modulo F and written as $h = NF_L(p, F)$.

If G is L -basis, then $NF_L(p, G)$ is uniquely defined ¹ for any polynomial p . In this case $NF_L(p, G) = 0$ if and only if p belongs to the ideal $\langle G \rangle$ generated by G . Moreover, if the ideal is *radical* for which any its element (polynomial) vanishes at the common roots of all the polynomials in G if and only if this polynomial belongs to the ideal, then it follows that the condition $NF_L(p, G) = 0$ is necessary and sufficient for vanishing p on those common roots.

It is important to emphasize that any involutive basis is a Gröbner basis, generally redundant, and can be used in the same manner as the reduced Gröbner basis [7, 8].

The above described and some other properties of the involutive bases as well as the Gröbner bases allow one to work fully algorithmically [18, 19] with constraints in the case of degenerated dynamical systems of polynomial type. In particular, one can work algorithmically on the constraint surface. In the next section considering the light-cone Yang-Mills mechanics we shall show how the above mentioned ideas of the involutive analysis can be realized in the computer algebra system *Maple*.

3 Application to Yang-Mills light-cone mechanics

Before demonstration of computer calculations we formulate the Yang-Mills light-cone mechanics. At first we use the standard Dirac-Hamilton formalism for systems with constraints. Then we explain some computational aspects of deriving the same results implementing the Dirac's method in terms of involutive polynomial bases [19] based on the Maple package.

3.1 Dirac's constrained dynamics

The Yang-Mills mechanics was formulated twenty years ago as instant form Yang-Mills field theory with spatially constant gauge fields and has been intensively studied during the last decades from different standpoints (see e.g [20]-[26] and references therein). The light-cone version of Yang-Mills mechanics is formulated analogously, it follows from the classical action functional for Yang-Mills field theory with an Ansatz that the gauge potential is a function of the light-cone time only. So, we start with the action for the pure Yang-Mills gauge field in four-dimensional Minkowski space M_4 , endowed with a metric η ²

$$I := \frac{1}{g^2} \int_{M_4} \text{tr} F \wedge *F, \quad (7)$$

¹For other properties of the involutive bases, proofs and illustrating examples see [4].

²In this paper we follow the notations of Ref. [27, 28]

where g is a coupling constant and the $su(2)$ algebra valued curvature two-form

$$F := dA + A \wedge A \quad (8)$$

is constructed from the connection one-form A . The connection and curvature, as Lie algebra valued quantities, are expressed in terms of the antihermitian $su(2)$ algebra basis $\tau^a = \sigma^a/2i$ with the Pauli matrices σ^a , $a = 1, 2, 3$,

$$A = A^a \tau^a, \quad F = F^a \tau^a. \quad (9)$$

The metric η enters the action through the dual field strength tensor defined in accordance to the Hodge star operation $*F_{\mu\nu} = \frac{1}{2} \sqrt{\eta} \epsilon_{\mu\nu\alpha\beta} F^{\alpha\beta}$. The light-cone coordinates $x^\mu = (x^+, x^-, x^\perp)$ are chosen as

$$x^\pm := \frac{1}{\sqrt{2}} (x^0 \pm x^3), \quad x^\perp := (x^k, k = 1, 2) \quad (10)$$

and the non-zero components of the metric η in the light-cone basis are

$$\eta_{+-} = \eta_{-+} = -\eta_{11} = -\eta_{22} = 1. \quad (11)$$

The connection one-form in the light-cone formulation is given as

$$A = A_+ dx^+ + A_- dx^- + A_k dx^k. \quad (12)$$

By definition the Lagrangian of the light-cone Yang-Mills mechanics follows from the corresponding Lagrangian of Yang-Mills theory if one supposes that connection one-form A depends on the light-cone “time variable” x^+ alone

$$A = A(x^+). \quad (13)$$

Using the definitions (7) and (12) we find the Lagrangian of the Yang-Mills light-cone mechanics

$$L := \frac{1}{2g^2} (F_{+-}^a F_{+-}^a + 2 F_{+k}^a F_{-k}^a - F_{12}^a F_{12}^a), \quad (14)$$

where the light-cone components of the field-strength tensor are given by

$$F_{+-}^a = \frac{\partial A_-^a}{\partial x^+} + \epsilon^{abc} A_+^b A_-^c, \quad (15)$$

$$F_{+k}^a = \frac{\partial A_k^a}{\partial x^+} + \epsilon^{abc} A_+^b A_k^c, \quad (16)$$

$$F_{-k}^a = \epsilon^{abc} A_-^b A_k^c, \quad (17)$$

$$F_{ij}^a = \epsilon^{abc} A_i^b A_j^c, \quad i, j, k = 1, 2. \quad (18)$$

Choosing the light-cone coordinate x^+ as an evolution parameter τ

$$\tau := x^+ \quad (19)$$

we define the light-cone version of the Hamiltonian description of a system with Lagrangian (14). The corresponding Hessian is degenerate, namely $\text{corank} \left\| \frac{\partial^2 L}{\partial \dot{A} \partial \dot{A}} \right\| = 6$, and thus the Legendre transformation ³

$$\pi_a^+ = \frac{\partial L}{\partial \dot{A}_+^a} = 0, \quad (20)$$

$$\pi_a^- = \frac{\partial L}{\partial \dot{A}_-^a} = \frac{1}{g^2} \left(\dot{A}_-^a + \epsilon^{abc} A_+^b A_-^c \right), \quad (21)$$

$$\pi_a^k = \frac{\partial L}{\partial \dot{A}_k^a} = \frac{1}{g^2} \epsilon^{abc} A_-^b A_k^c, \quad (22)$$

impose the six primary constraints on the canonical coordinates. Thus the generalized Hamiltonian system is characterized besides the canonical Poincaré-Cartan one-form

$$\Theta_C := \pi_a^+ dA_+^a + \pi_a^- dA_-^a + \pi_a^i dA_i^a - H_C d\tau \quad (23)$$

by set of primary constraints

$$\varphi_a^{(1)} := \pi_a^+ = 0, \quad (24)$$

$$\chi_k^a := g^2 \pi_k^a + \epsilon^{abc} A_-^b A_k^c = 0, \quad (25)$$

satisfying the following Poisson brackets relations

$$\{\varphi_a^{(1)}, \varphi_b^{(1)}\} = 0, \quad (26)$$

$$\{\varphi_a^{(1)}, \chi_k^b\} = 0, \quad (27)$$

$$\{\chi_i^a, \chi_j^b\} = -2 g^2 \epsilon^{abc} A_-^c \eta_{ij}. \quad (28)$$

In (23) the canonical light-front Hamiltonian is

$$H_C = \frac{g^2}{2} \pi_a^- \pi_a^- - \epsilon^{abc} A_+^b (A_-^c \pi_a^- + A_k^c \pi_a^k) + V(A_k), \quad (29)$$

where the potential term V is

$$V(A_k) = \frac{1}{2g^2} \left[(A_1^b A_1^b) (A_2^c A_2^c) - (A_1^b A_2^b) (A_1^c A_2^c) \right]. \quad (30)$$

Following the Dirac formulation, the dynamics of the light-cone Yang-Mills mechanics is governed by the total Hamiltonian

$$H_T = H_C - 2 \text{tr} (U(\tau) \varphi^{(1)}) - 2 \text{tr} (V_k(\tau) \chi_k), \quad (31)$$

³To simplify the formulas we shall use overdot to denote derivative of functions with respect to light-cone time τ . Further, we shall treat in equal footing the up and down isotopic indexes denoted with a, b, c, d .

where $U(\tau)$ and $V_k(\tau)$ are arbitrary $SU(2)$ valued functions of the light-cone time τ . Using this Hamiltonian we find that there are three secondary constraints $\varphi_a^{(2)}$

$$\varphi_a^{(2)} := \epsilon_{abc} (A_-^b \pi_c^- + A_k^b \pi_c^k) = 0, \quad (32)$$

obeying the $so(3, \mathbb{R})$ algebra

$$\{\varphi_a^{(2)}, \varphi_b^{(2)}\} = \epsilon_{abc} \varphi_c^{(2)}. \quad (33)$$

Checking analogously the time evolution of the primary constraints χ_k^a we have

$$0 = \dot{\chi}_k^a = \{\chi_k^a, H_C\} - 2g^2 \epsilon^{abc} V_k^b A_-^c. \quad (34)$$

The analysis of this equation depends on the properties of the matrix $\mathcal{C}_{ab} = \epsilon^{abc} A_-^c$. This matrix is degenerate with a rank vary from 0 to 2 depending on the point of the configuration space. If its rank is 2 then among the six primary constraints χ_k^a there are two first class constraints and maximum four Lagrange multipliers V can be determined from (34). When the rank of the matrix \mathcal{C}_{ab} is minimal, the locus points are $A_-^a = 0$ and all six constraints χ_k^a are Abelian ones. For such an exceptional configuration the constrained system reduces to dynamically trivial one and hereinafter we shall consider the subspace of configuration space where $\text{rank}||\mathcal{C}|| = 2$. For such configurations we are able to introduce the unit vector

$$N^a = \frac{A_-^a}{\sqrt{(A_-^1)^2 + (A_-^2)^2 + (A_-^3)^2}}, \quad (35)$$

that is a null vector of the matrix $||\epsilon^{abc} A_-^c||$, and to rewrite the six primary constraints χ_k^a as

$$\chi_{k\perp}^a := \chi_k^a - (N^b \chi_k^b) N^a, \quad (36)$$

$$\psi_k := N^a \chi_k^a. \quad (37)$$

Constraints $\chi_{k\perp}^a$ are functionally dependent due to the conditions

$$N^a \chi_{k\perp}^a = 0 \quad (38)$$

and choosing among them any four independent constraints, we are able to determine four Lagrange multipliers $V_{b\perp}^k$. The two constraints ψ_k satisfy the Abelian algebra

$$\{\psi_i, \psi_j\} = 0. \quad (39)$$

The Poisson brackets of the constraints ψ_k and $\varphi_a^{(2)}$ with the total Hamiltonian vanish after projection on the constraint surface (CS)

$$\{\psi_k, H_T\}|_{CS} = 0, \quad (40)$$

$$\{\varphi_a^{(2)}, H_T\}|_{CS} = 0 \quad (41)$$

and thus there are no ternary constraints.

Summarizing, we arrive at the set of constraints $\varphi_a^{(1)}, \psi_k, \varphi_a^{(2)}, \chi_{k\perp}^b$. The Poisson bracket algebra of the three first ones is

$$\{\varphi_a^{(1)}, \varphi_a^{(1)}\} = 0, \quad (42)$$

$$\{\psi_i, \psi_j\} = 0, \quad (43)$$

$$\{\varphi_a^{(2)}, \varphi_b^{(2)}\} = \epsilon_{abc} \varphi_c^{(2)}, \quad (44)$$

$$\{\varphi_a^{(1)}, \psi_k\} = \{\varphi_a^{(1)}, \varphi_b^{(2)}\} = \{\psi_k, \varphi_a^{(2)}\} = 0. \quad (45)$$

The constraints $\chi_{k\perp}^b$ satisfy the relations

$$\{\chi_{i\perp}^a, \chi_{j\perp}^b\} = -2g^2 \epsilon^{abc} A_-^c \eta_{ij}, \quad (46)$$

and the Poisson brackets between these two sets of constraints are

$$\{\varphi_a^{(2)}, \chi_{k\perp}^b\} = \epsilon^{abc} \chi_{k\perp}^c, \quad (47)$$

$$\{\varphi_a^{(1)}, \chi_{k\perp}^b\} = \{\psi_i, \chi_{j\perp}^b\} = 0. \quad (48)$$

From these relations we conclude that we have 8 first-class constraints $\varphi_a^{(1)}, \psi_k, \varphi_a^{(2)}$ and 4 second-class constraints $\chi_{k\perp}^a$. This means that now constraints reduce the 24 constrained phase space degrees of freedom to $24 - 4 - 2(5 + 3) = 4$ unconstrained degrees of freedom, in contrast to the instant form of the Yang-Mills mechanics where the number of the unconstrained canonical variables is 12.

3.2 Computational aspects of Dirac-Gröbner algorithm

Now we shall discuss what kind of computer algebra manipulations are necessary to perform in order to obtain the above stated results. We shall follow the general algorithm [18] adapted to the computer algebra manipulations in theories with polynomial Lagrangians. This algorithm, called Dirac-Gröbner algorithm, combines the constructive ideas of Dirac formalism for constrained systems with the Gröbner bases techniques.

Denote by q_α and \dot{q}_α , $1 \leq \alpha \leq 12$, respectively, the generalized Lagrangian coordinates in (14) listed as

$$A_+^1, A_+^2, A_+^3, A_1^1, A_1^2, A_1^3, A_2^1, A_2^2, A_2^3, A_-^1, A_-^2, A_-^3 \quad (49)$$

and their velocities (time derivatives). Then the momenta are $p_\alpha = \frac{\partial L}{\partial \dot{q}_\alpha}$, $1 \leq \alpha \leq 12$. To compute the primary constraints it suffices to eliminate the velocities \dot{q}_α from these system treated as polynomial in $\dot{q}_\alpha, q_\alpha, p_\alpha$. The elimination is performed by computing the Gröbner basis [7, 8] for the generating polynomial set

$$\left\{ p_\alpha - \frac{\partial L}{\partial \dot{q}_\alpha} \mid 1 \leq \alpha \leq 12 \right\} \quad (50)$$

for an ordering (in Maple `lexdeg`) eliminating velocities \dot{q}_α . In the obtained set all algebraically dependent constraints [8] are ruled out.

The canonical Hamiltonian (29) is determined as reduction of

$$p_\alpha \dot{q}_\alpha - L \quad (51)$$

modulo the Gröbner basis computed. Then computation of the Poisson brackets between the constraints is straightforward.

Next step is the construction of the secondary constraints (32). To find them we reduce the Poisson brackets of the primary constraints with the total Hamiltonian modulo the set of primary constraints. Again the Gröbner basis technique provides the right algorithmic tool for doing such computations and to obtain a complete set of twelve algebraically independent constraints. In order to compare with the constraints given in the previous section, we represent them as

$$\mathcal{F} = \{ \varphi_a^{(1)}, \psi_k, \varphi_a^{(2)}, \chi_k^1, \chi_k^2 \}, \quad a, b = 1, 2, 3; \quad k = 1, 2, \quad (52)$$

Next, to separate the complete set of constraints into first and second classes we compute the 12×12 Poisson bracket matrix on the constraint surface

$$M := \| \{f_\alpha, f_\beta\}_{|CS} \|, \quad \alpha, \beta = 1, \dots, 12, \quad (53)$$

where $f_\alpha, f_\beta \in \mathcal{F}$. Since the $\text{rank} \|M\| = 4$ the complete constraint set \mathcal{F} can be separated to four second-class constraints and eight first-class ones. To select the first-class constraints it suffices to compute the basis $\mathcal{A} = \{\mathbf{a}_1, \dots, \mathbf{a}_8\}$ of the null space of the matrix $\|M\|$ and then construct the first-class constraints as

$$\text{First class constraints} = \sum_{\alpha=1}^{12} (\mathbf{a}_i)_\alpha f_\alpha, \quad 1 \leq i \leq 8. \quad (54)$$

To extract the second-class constraints from \mathcal{F} one constructs 8×12 matrix $\|(\mathbf{a}_i)_\alpha\|$ from the components of the vectors in \mathcal{A} and find the basis $\mathcal{B} = \{\mathbf{b}_1, \dots, \mathbf{b}_4\}$ of the null space of the constructed matrix. Then every vector $\mathbf{b} \in \mathcal{B}$ yields a second-class constraint

$$\text{Second class constraints} = \sum_{\alpha=1}^{12} (\mathbf{b}_i)_\alpha f_\alpha, \quad 1 \leq i \leq 4. \quad (55)$$

As a result, one can organize the eight first-class constraints as $\varphi_a^{(1)}, \psi_k, \varphi_a^{(2)}$, whereas the four algebraically independent constraints χ_k^1, χ_k^2 are second-class.

Relations (42)-(45) revealing the structure of the gauge group generated by the first class constraints can also be computed fully algorithmically. To do this we extended the Maple package [18] with a general procedure to represent the Poisson brackets of any two first-class constraints f_α and f_β as a linear combination of elements in the set of first-class constraints:

$$\{f_\alpha, f_\beta\} = c^\gamma_{\alpha\beta} f_\gamma. \quad (56)$$

With that end in view and in order to cope the most general case we implemented the extended Gröbner basis algorithm [7]. Given a set of polynomials $Q = \{q_1, \dots, q_m\}$

generating the polynomial ideal $\langle Q \rangle$, this algorithm outputs the explicit representation

$$g_\alpha = h_{\alpha\beta} q_\beta \quad (57)$$

of elements in the Gröbner basis $G = \{g_1, \dots, g_n\}$ of this ideal in terms of the polynomials in Q . Having computed the Gröbner basis G for the ideal generated by the first-class constraints and the corresponding polynomial coefficients $h_{\alpha\beta}$ for the elements in G as given in (57), the local group coefficients $c_{\alpha\beta}^\gamma$ (which may depend on the generalized coordinates and momenta) in (56) are easily computed by reduction [7, 8] of the Poisson brackets modulo Gröbner basis expressed in terms of the first-class constraints.

However, the use of this universal approach may be very expensive from the computational point of view. For this reason our Maple package tries first to apply the multivariate polynomial division algorithm [8] modulo the set of first-class constraints. Due to the special structure of the primary first-class constraints that usually include those linear in momenta as in (24), this algorithm often produces the right representation (57); but unlike the extended Gröbner basis algorithm does it very fast. Correctness of the output is easily verified by computing of the reminder. If the last vanishes, then the output of the division algorithm is correct. Otherwise the extended Gröbner basis algorithm is applied.

In our case the division algorithm just produces the correct formulas (42)-(45) for the Poisson brackets of the first-class constraints $\varphi_a^{(1)}, \psi_k, \varphi_a^{(2)}$. Similarly, one obtains the formulas (46)-(48).

4 The unconstrained system as conformal mechanics

Now following the paper [27] we demonstrate how using Hamiltonian reduction of the degrees of freedom one can derive the unconstrained version of the light-cone $SU(2)$ Yang-Mills mechanics which coincides with the well-known model, the so-called conformal mechanics [29].

To show this equivalence we rewrite the model in terms of special coordinates as follows. At first we organize the configuration variables A_i^a and A_-^a in 3×3 matrix A_{ab} whose entries of the first two columns are A_i^a and the third column is composed by the elements A_-^a

$$A_{ab} := \|A_1^a, A_2^a, A_-^a\|. \quad (58)$$

In order to find an explicit parameterization of the orbits and the slice structure with respect to the gauge symmetry action, it is convenient to use a polar decomposition for the matrix A_{ab}

$$A = OS, \quad (59)$$

where S is a positive definite 3×3 symmetric matrix, $O(\phi_1, \phi_2, \phi_3) = e^{\phi_1 J_3} e^{\phi_2 J_1} e^{\phi_3 J_3}$ is an orthogonal matrix parameterized by the three Euler angles (ϕ_1, ϕ_2, ϕ_3) and $SO(3, \mathbb{R})$ generators in adjoint representation $(J_a)_{ij} = \epsilon_{iaj}$. Here we assume that the matrix A has a positive determinant and hence treat the polar decomposition (59) as a uniquely invertible

transformation from the configuration variables A_{ab} to a new set of Lagrangian variables: six coordinates S_{ij} and three coordinates ϕ_i .⁴ The polar decomposition (59) induces the point canonical transformation from the coordinates A_{ab} and $\Pi_{ab} := \|\pi^{a1}, \pi^{a2}, \pi^{a-}\|$ to new canonical pairs (S_{ab}, P_{ab}) and (ϕ_a, P_a) with the following non-vanishing Poisson brackets

$$\{S_{ab}, P_{cd}\} = \frac{1}{2} (\delta_{ac} \delta_{bd} + \delta_{ad} \delta_{bc}) , \quad (60)$$

$$\{\phi_a, P_b\} = \delta_{ab} . \quad (61)$$

The expression of the old Π_{ab} as a function of the new coordinates is

$$\Pi = O (P - k_a J_a) , \quad (62)$$

where

$$k_a = \gamma_{ab}^{-1} (\eta_b^L - \varepsilon_{bmn} (SP)_{mn}) , \quad (63)$$

$\gamma_{ik} = S_{ik} - \delta_{ik} \text{tr} S$ and η_a^L are three left-invariant vector fields on $SO(3, \mathbb{R})$ group (the explicit form of η_a^L in terms of the angular variables ϕ_a, P_a can be found in [27]). In terms of the new variables the constraints $\varphi_a^{(2)}$ and χ_i^a can be rewritten in the equivalent form

$$\eta_a^L = 0 , \quad (64)$$

$$\tilde{\chi}_{ai} = P_{ai} + \epsilon_{aij} \gamma_{jk}^{-1} \epsilon_{kmn} (SP)_{mn} + \epsilon_{amn} S_{m3} S_{ni} = 0 \quad (65)$$

with vanishing Poisson brackets

$$\{\eta_a^L, \tilde{\chi}_{bi}\} = 0 . \quad (66)$$

Passing to the new polar decomposition variables we achieve the complete separation of variables (S_{ab}, P_{cd}) , which are invariant under the gauge transformations generated by Gauss law constraints $\varphi_a^{(2)}$, from the gauge variant one (ϕ_a, P_a) . Owing to the separation to eliminate all gauge degrees of freedom related to this symmetry it is enough to project to the constraint shell described by the condition of nullity of Killing fields η_a^L . The corresponding pure gauge degrees of freedom will automatically disappear from the projected Hamiltonian.

In order to proceed further in elimination of remaining constraints we introduce the main-axes decomposition for the symmetric 3×3 matrix S

$$S = R^T(\chi_1, \chi_2, \chi_3) \begin{pmatrix} q_1 & 0 & 0 \\ 0 & q_2 & 0 \\ 0 & 0 & q_3 \end{pmatrix} R(\chi_1, \chi_2, \chi_3) , \quad (67)$$

⁴Note that the polar decomposition is valid for an arbitrary matrix but the orthogonal matrix in (59) is uniquely determined only from the invertible matrix $O = AS^{-1}$, $S = \sqrt{AA^T}$ and thus only in this case the polar decomposition (59) is a well-defined coordinate transformation. Concerning the degenerate matrices the more close and subtle analysis should be done. Here we note only that the set of $n \times n$ matrices with rank k is a manifold with dimension $k(2n - k)$, but as distinct from the non-degenerate case now the manifold atlas contains with a necessity several charts.

with an orthogonal matrix R , parameterized by three Euler angles χ_1, χ_2, χ_3 . Because the Jacobian of this transformation is $J\left(\frac{S[q, \chi]}{q, \chi}\right) \sim \prod_{a \neq b}^3 |q_a - q_b|$ the equation (67) can be used as definition of new configuration variables: three “diagonal” variables (q_1, q_2, q_3) , eigenvalues of the matrix S , and three angular variables (χ_1, χ_2, χ_3) , if and only if all eigenvalues of the matrix S are different. To have the uniqueness of the inverse transformation we assume here that $q_1 < q_2 < q_3$.⁵

The momenta p_a and p_{χ_a} , canonically conjugated to the diagonal and angular variables q_a and χ_a , can be found using the condition of the canonical invariance of the symplectic one-form

$$\sum_{a,b=1}^3 P_{ab} dS_{ab} = \sum_{a=1}^3 p_a dq_a + \sum_{a=1}^3 p_{\chi_a} d\chi_a. \quad (68)$$

The original momenta P_{ab} , expressed in terms of the new canonical variables, read

$$P = R^T \sum_{a=1}^3 \left(p_a \bar{\alpha}_a - \frac{1}{2} \frac{\xi_a^R}{q_b - q_c} \alpha_a \right) R, \quad (\text{cyclic permutations } a \neq b \neq c). \quad (69)$$

Here $\bar{\alpha}_a$ and α_a denote the diagonal and off-diagonal basis elements for the space of symmetric matrices which obey the relations $\text{tr}(\bar{\alpha}_a \bar{\alpha}_b) = \delta_{ab}$, $\text{tr}(\alpha_a \alpha_b) = 2\delta_{ab}$, $\text{tr}(\bar{\alpha}_a \alpha_b) = 0$.

The ξ_a^R are three $SO(3, \mathbb{R})$ right-invariant vector fields given in terms of the angles χ_a and their conjugated momenta p_{χ_a} via

$$\xi_1^R = -\sin \chi_1 \cot \chi_2 p_{\chi_1} + \cos \chi_1 p_{\chi_2} + \frac{\sin \chi_1}{\sin \chi_2} p_{\chi_3}, \quad (70)$$

$$\xi_2^R = \cos \chi_1 \cot \chi_2 p_{\chi_1} + \sin \chi_1 p_{\chi_2} - \frac{\cos \chi_1}{\sin \chi_2} p_{\chi_3}, \quad (71)$$

$$\xi_3^R = p_{\chi_1}. \quad (72)$$

The constraints $\tilde{\chi}$ rewritten in the main-axes variables take the form

$$\tilde{\chi} = \sum_{a=1}^3 R^T \left[\pi_a \bar{\alpha}_a - \frac{1}{2} \rho_a^- \alpha_a + \frac{1}{2} \rho_a^+ J_a \right] R, \quad (73)$$

where

$$\rho_a^\pm = \frac{\xi_a^R}{q_b \pm q_c} \pm \frac{1}{g^2} q_a n_a (q_b \pm q_c), \quad (74)$$

and $n_a = R_{a3}$.

⁵The variables q_a in the main-axes transformation (67) parameterize the orbits of the adjoint action of $SO(3, \mathbb{R})$ group in the space of 3×3 symmetric matrices. Whereas the consideration of the configuration $q_1 < q_2 < q_3$ describing the so-called principle orbit class given below is correct, the treatment of all orbits with coinciding eigenvalues of the matrix S , the singular orbits, requires more skillful treatment that is beyond the scope of the present paper.

As it was shown above the constraints χ_i^a represent a mixed system of first and second class constraints, ψ_i and $\chi_{i\perp}^a$ correspondingly. To perform the reduction to the constraint shell it is useful at first to introduce a gauge fixing condition and eliminate the two first class constraints ψ_i . The expression (37) for the Abelian constraints ψ_i dictates the appropriate gauge fixing condition

$$\overline{\psi}_i := N^a A_i^a = 0, \quad (75)$$

which is canonical one in the sense that $\{\overline{\psi}_i, \psi_j\} = \delta_{ij}$. The constraints $\psi_i = 0$ together with the canonical gauge-fixing condition (75) rewritten in terms of the main-axis variables fixes the canonical angular variables

$$\chi_1 = 0, \quad p_{\chi_1} = 0, \quad \chi_2 = \frac{\pi}{2}, \quad p_{\chi_2} = 0. \quad (76)$$

Examine the remaining four second class constraints χ_i^1 and χ_i^2 in terms of the main-axes variables we find that the corresponding constraint shell can be described by the following conditions on the “diagonal” canonical pairs

$$p_1 = 0, \quad p_3 = 0, \quad (q_1 \pm q_3)^2 = \pm g^2 \frac{\xi_2^R}{q_2}. \quad (77)$$

Now using all above expressions for the constraints one can easily project the total Hamiltonian (31) on the constraint shell and convince that the dynamics of the two unconstrained canonical pairs (q_2, p_2) and (χ_3, p_{χ_3}) is governed by the following reduced Hamiltonian

$$H_{LC}^* = \frac{g^2}{2} \left(p_2^2 + \frac{\alpha}{q_2^2} \right), \quad (78)$$

where $\alpha = \frac{p_{\chi_3}^2}{4}$.⁶ Because the momentum p_{χ_3} is conserved one can identify the reduced Hamiltonian of $SU(2)$ light-cone mechanics with the Hamiltonian of conformal mechanics whose “coupling constant” is determined by the value of α , while the gauge field coupling constant g controls the scale for the evolution parameter.

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⁶Here we use that the expression for ξ_2^R reduces to $-p_{\chi_3}$ on the constraint shell.

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